# An asymptotic criterion in an explicit sequence

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**André Voros**
CEA-Institut de Physique Théorique de Saclay (CNRS UMR 3681)
F-91191 Gif-sur-Yvette Cedex (France)
E-mail: andre.voros@cea.fr

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**Abstract**

We report a novel asymptotic (large-order) behavior in an explicit sequence built out of the Bernoulli numbers and analyzed by a variant of instanton calculus or Darboux's theorem.

We will use: $B_{2m}$ : the Bernoulli numbers; $\gamma$ : Euler's constant; $k!! = k(k-2)(k-4)\ldots$ : double factorial (with $0!! = (-1)!! = 1$ as usual).

The real sequence explicitly spelled out for $n = 1, 2, \ldots$ as

$$u_n = (-1)^n \Bigg[ 2^{-2n} \sum_{m=1}^{n} \frac{(-1)^m}{2m-1} \binom{2(n+m)}{n+m} \binom{n+m}{2m} \log \frac{|B_{2m}|}{(2m-3)!!} - \frac{(2n)!!}{2\,(2n-1)!!} \log 2\pi \Bigg] \quad (1)$$

can be thus numerically computed (trivially to thousands of terms), and very early it satisfies (figs.)

$$u_n \approx \log n - 1.703\ . \quad (2)$$

This can be validated *assuming the Riemann Hypothesis* (= RH), as

$$u_n \sim \log n + K, \qquad K = \tfrac{1}{2}(\gamma - \log(2\pi^2) - 1) \approx -1.70269564368\,. \quad (3)$$

With RH verified up to an ordinate $T_0 \gtrsim 2 \cdot 10^{12}$ currently, [5] it is plausible indeed to witness the behavior (3).

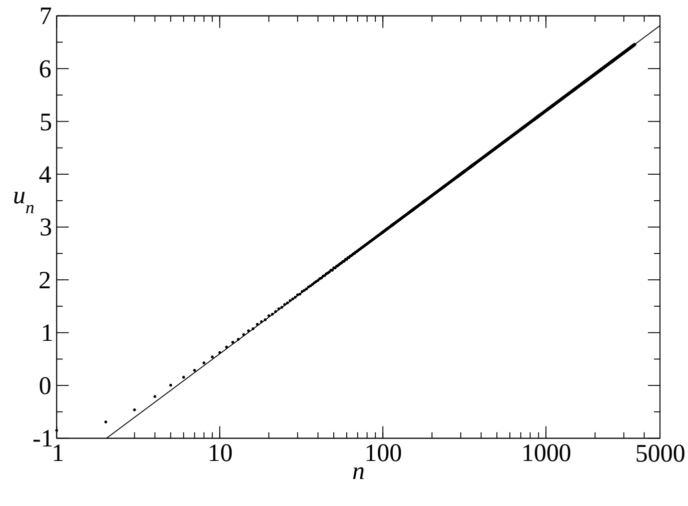


Figure 1: The coefficients $u_n$ computed by (1) up to $n = 3500$, on a logarithmic $n$-scale, vs the function $(\log n + K)$ of (3) (straight line); the first values are $u_1 = \log\pi - \frac{1}{2}\log 54 \approx -0.84976213743$, $u_2 = -\frac{4}{3}\log\pi + \frac{23}{24}\log 2 + \frac{55}{24}\log 3 - \frac{35}{24}\log 5 \approx -0.69148426053$, $u_3 \approx -0.46222439972$.

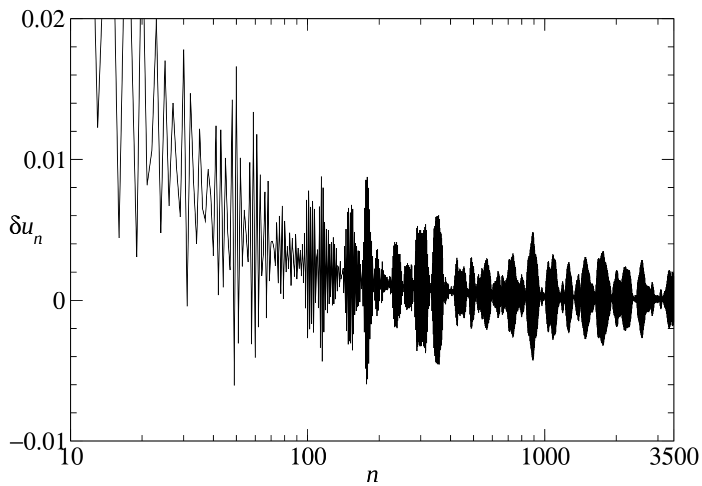


Figure 2: As fig. 1 but for the remainders $\delta u_n = u_n - (\log n + K)$, on a very dilated vertical scale. (The connecting segments between data points are only drawn for clarity.)

Now by large-order analysis through exponential asymptotics, [3][1] we found that *if RH is false*, $u_n$ will also admit a clear-cut asymptotic form but of a wholly different nature, dominated by individual terms $F_n(\rho)$ contributed by every zero $\rho = \frac{1}{2} + t + \mathrm{i}T$ off the critical line with $0 < t$ $(< \frac{1}{2})$ :

$$F_n(\rho) \sim f(\rho)(-1)^n \frac{(2n)^{\rho-1/2}}{\log n} \qquad \text{for } n \to \infty, \tag{4}$$

where $f$ is an explicit function independent of $n$ with the main property

$$|f(\rho)| \approx |T|^{-t-2} \text{ for } |T| \gg 1 \quad \Longrightarrow \quad |F_n(\rho)| \approx |T|^{-t-2}(2n)^t/\log n. \tag{5}$$

Each such $F_n(\rho)$ will ultimately dominate (3) in the $n \to \infty$ limit, but starts out exceedingly tiny at low $n$, and does not approach unity until

$$n \gtrsim \tfrac{1}{2}|T|^{1+2/t} \qquad (\text{at best } O(|T|^{5+\varepsilon}) \text{ for } t \to \tfrac{1}{2}^-); \tag{6}$$

yet we think that, with efficient signal-processing, the "signal" $F_n(\rho)$ of $\rho$ within $u_n$ ought to be detectable much sooner than at (6) (at $n \gtrsim O(|T|^{1+1/t})$ or even less, but within $n \gg |T|$). Still, to seek a violation of RH and verify the form (4), $|T| > T_0$ is necessary, and $T_0 \gtrsim 2 \cdot 10^{12}$ implies very large $n$-values.

The major issue is then that $u_n$ is an alternating sum of terms which turn out to be exponentially larger by an order of $(3 + 2\sqrt{2})^n$. Increasingly delicate cancellations thus take place, requiring a precision beyond $\approx 0.7656\,n$ decimal digits to evaluate $u_n$ by (1). On the other hand, this purely technical demand seems to be the *sole* obstacle raised by the use of (1) at unlimited $n$.

While other sequences sensitive to RH for large $n$ are known, [6][2][7][4] we are unaware of any previous case combining a fully *closed form* like (1) with a practical sensitivity threshold of *tempered growth* $n = O(T^\nu)$.

Details and derivations are currently under completion. [8]

# References


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